CFA Institute

QUANTITATIVE METHODS

CFA® Program Curriculum

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How to Use the CFA Program Curriculum

The CFA® Program exams measure your mastery of the core knowledge, skills, and abilities required to succeed as an investment professional. These core competencies are the basis for the Candidate Body of Knowledge (CBOK™). The CBOK consists of four components:

A broad outline that lists the major CFA Program topic areas (www.cfainstitute.org/programs/cfa/curriculum/cbok/cbok)

Topic area weights that indicate the relative exam weightings of the top-level topic areas (www.cfainstitute.org/en/programs/cfa/curriculum)

Learning outcome statements (LOS) that tell you the specific knowledge, skills, and abilities you should gain from each curriculum topic area. You will find these statements at the start of each learning module and lesson. We encourage you to review the information about the LOS on our website (www.cfainstitute.org/programs/cfa/curriculum/study-sessions), including the descriptions of LOS "command words" on the candidate resources page at www.cfainstitute.org/-/media/documents/support/programs/cfa-and-cipm-los-command-words.ashx.

The CFA Program curriculum that candidates receive access to upon exam registration.

Therefore, the key to your success on the CFA exams is studying and understanding the CBOK. You can learn more about the CBOK on our website: www.cfainstitute.org/programs/cfa/curriculum/cbok.

The curriculum, including the practice questions, is the basis for all exam questions. The curriculum is selected/developed specifically to provide candidates with the knowledge, skills, and abilities reflected in the CBOK.

CFA INSTITUTE LEARNING ECOSYSTEM (LES)

Your exam registration fee includes access to the CFA Institute Learning Ecosystem (LES). This digital learning platform provides access to all the curriculum content and practice questions. The LES is organized as a series of learning modules consisting of short online lessons and associated practice questions. This tool is your source for all study materials, including practice questions and mock exams. The LES is the primary method by which CFA Institute delivers your curriculum experience. Here, you will find additional practice questions to test your knowledge, including some interactive questions.

DESIGNING YOUR PERSONAL STUDY PROGRAM

An orderly, systematic approach to exam preparation is critical. You should dedicate a consistent block of time every week to reading and studying. Review the LOS both before and after you study curriculum content to ensure you can demonstrate the knowledge, skills, and abilities described by the LOS and the assigned learning module. Use the LOS as a self-check to track your progress and highlight areas of weakness for later review.

Successful candidates report an average of more than 300 hours preparing for each exam. Your preparation time will vary based on your prior education and experience, and you will likely spend more time on some topics than on others.

ERRATA

The curriculum development process is rigorous and involves multiple rounds of reviews by content experts. Despite our efforts to produce a curriculum that is free of errors, we must make corrections in some instances. Curriculum errata are periodically updated and posted by exam level and test date on the Curriculum Errata webpage (www.cfainstitute.org/en/programs/submit-errata). If you believe you have found an error in the curriculum, you can submit your concerns through our curriculum errata reporting process found at the bottom of the Curriculum Errata webpage.

OTHER FEEDBACK

Please send any comments or suggestions to info@cfainstitute.org, and we will review your feedback thoughtfully.

Quantitative Methods

LEARNING MODULE

1

Returns of Financial Assets and Instruments

LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	describe, compare, and interpret returns
☐	describe, compare, and interpret required rates of return, risk-free rates, risk premia, and inflation

INTRODUCTION: RETURNS OF FINANCIAL ASSETS AND INSTRUMENTS

1

The Quantitative Methods learning modules focus on the fundamental building blocks that shape investment decision-making and investment strategies. The first learning module introduces the calculation and interpretation of returns and rates. It then discusses different return measures and demonstrates their calculation approaches.

Subsequent learning modules introduce other key quantitative building blocks: time value of money, statistical properties of financial and economic data, portfolio mathematics, modern portfolio theory, regression analysis, and asset pricing. Practical market examples support these building blocks as they will reappear throughout the three levels of the curriculum.

PRE-TEST: RETURNS OF FINANCIAL ASSETS AND INSTRUMENTS

1. SolarTech Innovations' shares showed the following total returns:

   - Year 1: −10%
   - Year 2: 15%
   - Year 3: 20%

   Over the three-year period, SolarTech shares had a geometric annual return closest to:

   A. 7.49%.

B. 8.33%.

C. 14.93%.

Solution

The correct response is A, 7.49%. Geometric mean return accurately captures the compound interest effect over multiple periods:

$$[(1-0.10)\times(1+0.15)\times(1+0.20)]^{\frac{1}{3}}-1=7.49\%.$$

This method accounts for the compounding effect of gains and losses over the years.

Answer B is incorrect. It computes the arithmetic average:

$$\frac{-10\%+15\%+20\%}{3}=8.33\%.$$

Answer C is incorrect. It treats the negative return during the first year incorrectly:

$$[(1+0.10)\times(1+0.15)\times(1+0.20)]^{\frac{1}{3}}-1=14.93\%.$$

2. A stock experiences a continuously compounded return of 10% for the first half of a year and 40% for the second half. Its continuously compounded return for the full year is closest to:

A. 50.00%.

B. 54.00%.

C. 56.25%.

Solution

The correct response is A, 50.00%. Contiuously compounded returns are additive over time. For each period's return, we directly add them: 10% for the first half and 40% for the second half, 10% + 40% = 50%.

Answer B is incorrect. It does not consider the additive property of continuously compounded returns: 1.1 × 1.4 – 1 = 54%. This treats the returns as if they were realized simple returns for the period and ignores that they are continuously compounded.

Answer C is incorrect. It first averages the continuously compounded returns, $\frac{10\%+40\%}{2}=25\%$, before calculating the period returns, $1.25^2-1=$ 56.25%.

3. An investor purchases 1,000 shares at USD100 per share. After she receives a dividend of USD3 per share one year later, she sells the shares for USD120 per share. Her tax rate is 25% on capital gains and 15% on capital distributions. Her after-tax return for the year is closest to:

A. 17.25%.

B. 17.55%.

C. 19.25%.

Solution

The correct response is B, 17.55%. The after-tax return can be calculated using the following relationship:

$$r_{net-tax}=r_{price}\times(1-tax_{capitalgains})+r_{distributions}\times(1-tax_{distributions}).$$

First, the price return is calculated:

$$r_{price} = \frac{P_1 - P_0}{P_0} = \frac{\text{USD120} - \text{USD100}}{\text{USD100}} = 20.0\%.$$

Then, the capital distribution return is calculated:

$$r_{distributions} = \frac{Inc}{P_0} = \frac{\text{USD3}}{\text{USD100}} = 3.0\%.$$

Finally, the respective after-tax returns are calculated:

$$r_{price} \times (1 - tax_{capital\ gains}) + r_{distributions} \times (1 - tax_{distributions})$$

$$= 20.0\% \times (1 - 25\%) + 3.0\% \times (1 - 15\%) = 15.0\% + 2.55\% = 17.55\%.$$

Answer A is incorrect. It applies the higher capital gains tax rate on both sources of returns:

$$r_{price} \times (1 - tax_{capital\ distributions}) + r_{distributions} \times (1 - tax_{capital\ gains})$$

$$= 20.0\% \times (1 - 25\%) + 3.0\% \times (1 - 25\%) = 15.0\% + 2.25\% = 17.25\%.$$

Answer C is incorrect. It applies incorrect tax rates; specifically, it reverses the rates.

$$r_{price} \times \left(1 - tax_{capital\ distributions}\right) + r_{distributions} \times \left(1 - tax_{capital\ gains}\right)$$

$$= 20.0\% \times (1 - 15\%) + 3.0\% \times (1 - 25\%) = 17.0\% + 2.25\% = 19.25\%.$$

4. In investment decisions, the risk premium:

   A. guarantees that expected returns meet actual returns.

   B. compensates investors for uncertainty associated with returns.

   C. ensures that risk-free investments deliver consistent returns over time.

Solution

The correct answer is B. The risk premium compensates for the risk associated with the variance between expected, *ex ante*, and actual, *ex post*, returns. It compensates for the uncertainty between expected and realized returns. The size of the compensation reflects the increased uncertainty about expected returns: The higher the uncertainty, the higher the risk premium.

Answer A is incorrect. The risk premium cannot guarantee that expected returns will always meet realized returns. The risk premium compensates for the inherent uncertainty between expectations and outcomes.

Answer C is incorrect. The risk premium is not intended to make risk-free investments deliver consistent returns over time. Risk-free investments do not face default or reinvestment risk.

FINANCIAL RETURNS 2

☐ describe, compare, and interpret returns

Returns are a critical component of understanding the financial markets, yet returns cannot be distilled into one easily defined concept. The different types of **financial assets** and instruments bundle series of cash flows that provide distinctly different returns to investors. This first learning module provides a brief overview of the types of returns provided by financial assets and instruments. The next learning module will discuss the source of their returns and their historical performance.

- Financial assets include cash, such as coins and bills; equity or ownership interest; debt, a liability of the debtor to the creditor or investor, which is an asset of the creditor or investor; and hybrid securities that combine these characteristics. These assets represent various forms of value held by individuals or entities and play crucial roles in investment and financing activities.
- Financial instruments include equity and debt and different types of standardized and tradable contractual relationships that reflect value, such as rights or derivatives. Both debt and equity are well-defined, highly standardized financial assets that are traded as financial instruments. Financial instruments bundle different patterns of cash flows and payoffs together into a single security or contract. This standardization and tradability facilitate the efficient exchange of these instruments between investors in financial markets.
- **Financial indicators** are observable but not directly tradable indicators of value and include exchange rates, interest rates, and market indexes. Unlike financial assets and instruments, financial indicators do not generate cash flows.

The distinction between financial assets and financial instruments is nuanced. **Financial instruments** are the means through which financial assets are standardized, packaged, and traded in financial markets. Other alternative approaches exist to group and define investments.

This section covers the basics of return on debt and equity investments, which account for the bulk of investable assets. Although other asset classes, such as commodities, alternative assets, and digital assets, are not specifically covered, the building blocks described below are generally applicable to them as well.

Total Return

Financial assets, instruments, and indicators are frequently defined in terms of their return and risk characteristics. Returns are normally generated in two ways.

Capital appreciation return, or price return, is the change in the price between two different points of time, where P0 denotes the price when the initial investment is made and P1 represents the price at a later valuation date:

$$r_{price} = \frac{P_1 - P_0}{P_0}. \quad (1)$$

Capital distribution return is income through cash dividends, interest payments, or other types of capital distributions, denoted here as Inc and expressed in terms of the initial investment made in the asset:

$$r_{distribution} = \frac{Inc}{P_0}. \quad (2)$$

For capital distributions on equities, we focus on the dividend yield; for bonds, we look at current yield. **Dividend yield** is calculated as the annual dividends paid by a company divided by its share price at the beginning of the period, or P_0. **Current yield** is calculated as the annual coupon payments divided by the market price of the bond.

The **total return** encompasses both sources of return, capital appreciation and capital distributions, and is integral in assessing the performance of an investment:

$$r = r_{price} + r_{distribution}$$

$$= \frac{P_1 - P_0}{P_0} + \frac{Inc}{P_0}$$

$$= \frac{P_1 - P_0 + Inc}{P_0}. \quad (3)$$

Some financial assets provide return through only one of these mechanisms. Investors in non-dividend-paying stocks obtain return only from price movements. Investors in **annuity contracts**, a financial product that pays out a fixed stream of payments during the life of the contract, do not receive any other income or distributions.

Whether investors measure returns over a single period or over multiple periods, returns are expressed as decimals (0.09), fractions (9/100), and percentages (9.0%). This simplifies comparison across different investments and time periods. Calculating single period returns is straightforward, as they measure the change in an asset's value and capital distribution return over a single period: a day, a month, a quarter, or a year. Multi-period returns, assessing long-term value accumulation of capital relevant to investment management, are more complicated and will be covered later in this module.

Returns can be expected returns and actual returns. **Expected returns**, or ex ante returns, are the anticipated returns on an investment over a specified period. They are not guaranteed but are estimated based on historical data, current market conditions, and analysis of the asset or investment. Expected returns feed into the investment management and decision-making process, representing the investor's forecasts before the investment's outcome is known. **Actual returns**, or ex post returns, are the returns on an investment after the investment period has ended. They may differ from expected returns due to unforeseen market movements, changes in economic conditions, or specific events affecting the asset or market.

The concept of risk and risk premia, which we will discuss later, primarily revolves around compensating for the difference between expected returns, or ex ante, and actual returns, or ex post. This distinction highlights the inherent **risk** in investment decision making: Decisions are based on expected returns but ultimately deliver actual returns. The greater the likelihood that actual returns will fall short of expected returns, the greater the compensation investors will demand.

While an investor holds an investment, changes in value represent **unrealized returns**. These are gains and losses that would be realized if the investment were sold at the current market value. These unrealized profits are also called "paper gains." When the investment is ultimately sold, these returns become **realized returns**. Capital distributions, such as dividends or interest while the investment is held, are also realized returns that are generally recognized in the period in which they are received. In most cases, taxes are applied only when gains are realized.

The following case study analyzes the performance of Ørsted A/S, a renewable energy company based in Denmark.

CASE STUDY

Calculating Ørsted A/S Bond and Equity Returns

Ørsted A/S, www.orsted.com, a Danish company, operates in the renewable energy sector, including the development, construction, and operation of offshore and onshore wind farms, bioenergy plants, solar farms, energy storage facilities. It also engages in waste-to-energy solutions and smart energy products. The company's equity is traded on the NASDAQ Copenhagen, formerly known as the Copenhagen Stock Exchange, https://www.nasdaqomxnordic.com/. Exhibit 1 shows the long-term performance of the stock in a Bloomberg screenshot using the ORSTED DC Equity GP <GO> function (in Danish krone, DKK).

Exhibit 1: Ørsted A/S Equity Performance1 January 2016–29 December 2023 (in DKK)

Source: Bloomberg.

Exhibit 2 provides the year-end prices and dividends for Ørsted A/S shares from 2017–2023, with values denominated in DKK.

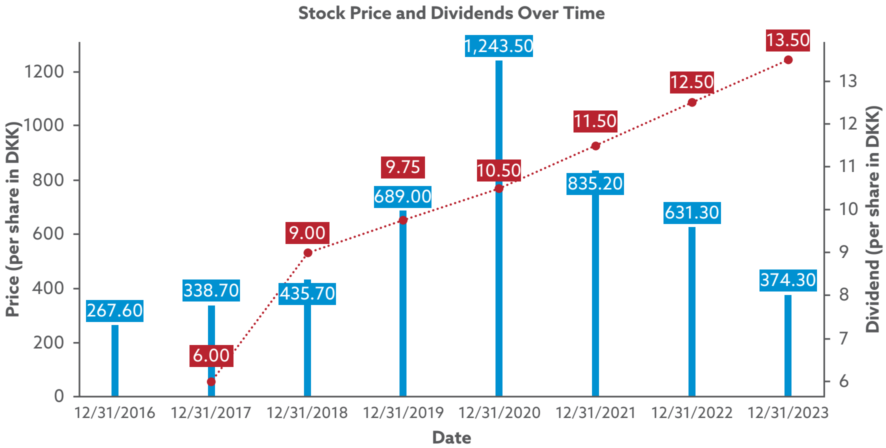
Exhibit 2: Ørsted A/S Equity Year-End Prices and Annual Dividends, 2017–2023 (in DKK)

Date	Price (per share in DKK)	Dividend (per share in DKK)
31 December 2016*	267.60	---
31 December 2017	338.70	6.00
31 December 2018	435.70	9.00
31 December 2019	689.00	9.75
31 December 2020	1,243.50	10.50
31 December 2021	835.20	11.50
31 December 2022	631.30	12.50
31 December 2023	374.30	13.50

**The 31 December 2016 price, or year-end price, is used as the starting value for the year 2017.*

Exhibit 3 shows the same information as that in Exhibit 2 graphically. It depicts the appreciation of the stock price from year end 2016–2020 and its subsequent decline from 2020 to 2023. At the same time, it depicts how dividends have increased steadily.

Exhibit 3: Ørsted A/S Equity Year-End Prices and Annual Dividends, 2016–2023 (in DKK)



Note: The 31 December 2016 price, or year-end price, is used as the starting value for the year 2017.

On 24 November 2017, Ørsted A/S entered the Green Bond market with its EUR750 million Senior Unsecured Bond listed on the London Stock Exchange. This bond, with a fixed annual coupon rate of 1.5%, is set to mature on 26 November 2029. It pays an annual coupon of EUR15 per EUR1,000 principal, with the first payment made on 18 November 2018. The bond's proceeds are allocated to finance offshore wind, bioenergy, and smart grid projects. These investments aim to facilitate a transition to a low-carbon, climate-resilient, and sustainable economy, potentially reducing emissions by 104,000 metric tons of CO2 per year. Exhibit 4 shows the long-term performance of this bond issue in a Bloomberg screenshot using the ORSTED 1 ½ 11/26/29 Corp GP <GO> function.

Exhibit 4: Ørsted A/S Bond Performance, 1 December 2017–29 December 2023 (in EUR)

Source: Bloomberg.

Exhibit 5 provides the year-end price of the bond from 2018–2023 and its annual coupon payments in EUR. Note that the coupon payments occurred earlier in the year, sometime in November.

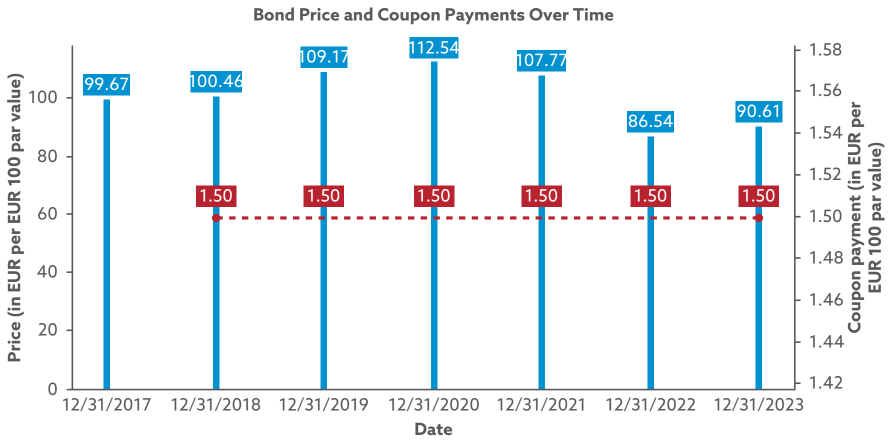
Exhibit 5: Ørsted A/S EUR750 Million Senior Unsecured Bond Year-End Prices and Annual Coupon Payments, 2017–2023 (in EUR)

Date	Price (in EUR per EUR 100 par value)	Annual Coupon Payment (in EUR per EUR 100 par value)
31 December 2017	99.672	---
31 December 2018	100.462	1.50
31 December 2019	109.173	1.50
31 December 2020	112.540	1.50
31 December 2021	107.769	1.50
31 December 2022	86.536	1.50
31 December 2023	90.614	1.50

Note: The 31 December 2017 price, or year-end price, is used as the starting value for the year 2018.

Exhibit 6 presents the same information as in Exhibit 5 graphically. It shows the variability in the price of the bond and the steady fixed annual coupon payment investors receive.

Exhibit 6: Ørsted A/S EUR750 Million Senior Unsecured Bond Year-End Prices and Annual Coupon Payments (in EUR)



Note: The 31 December 2017 price, or year-end price, is used as the starting value for the year 2018.

1. Calculate the price return, dividend yield, and total return, for Ørsted A/S equity in 2020.

   Solution

   In 2020, the equity of Ørsted A/S had a price return of 80.48% and a dividend yield of 1.52%, resulting in a total return of 82.00%. To arrive at this result, there are three steps: Calculate the price return, then determine the capital distribution return, and finally add these two returns together.

   1. Price return

The price return is the change in the price between two different points of time, where P0 denotes the price when the initial investment is made and P1 represents the price at a later valuation date. Here we need the price at the beginning of 2020 and the end of 2020. The price at the beginning of 2020 is the same as the price at the end of 2019, hence P0 is DKK689.00 and P1 is DKK1,243.50. Using Equation 1:

$$r_{price} = \frac{P_1 - P_0}{P_0} = \frac{\text{DKK1,243.50} - \text{DKK689.00}}{\text{DKK689.00}} = \frac{\text{DKK554.50}}{\text{DKK689.00}} = 80.48\%.$$

2. Capital distribution return

The dividend yield is the capital distribution return, income through cash dividends, denoted Inc, and expressed in terms of the initial investment made in the asset. The dividend for 2020 was DKK10.50. Substituting into Equation 2:

$$r_{distribution} = \frac{Inc}{P_0} = \frac{\text{DKK10.50}}{\text{DKK689.00}} = 1.52\%.$$

3. Total return

The total return includes both sources of return. Substituting into Equation 3:

$$r = r_{price} + r_{\text{distribution}} = \frac{P_1 - P_0}{P_0} + \frac{Inc}{P_0}$$

$$= \frac{\text{DKK1,243.50} - \text{DKK689.00}}{\text{DKK689.00}} + \frac{\text{DKK10.50}}{\text{DKK689.00}}$$

$$= \frac{\text{DKK1,243.50} - \text{DKK689.00} + \text{DKK10.50}}{\text{DKK689.00}}$$

$$= \frac{\text{DKK554.50} + \text{DKK10.50}}{\text{DKK689.00}} = \frac{\text{DKK565.00}}{\text{DKK689.00}} = 82.00\%.$$

2. Calculate the price return, current yield, and total return for Ørsted A/S in 2020 for its EUR750 million Senior Unsecured Bond listed on the London Stock Exchange.

Solution

In 2020, the EUR750 million Senior Unsecured Bond issued by Ørsted A/S had a price return of 3.08% and a current yield of 1.37%, resulting in a total return of 4.46%. Similar to the previous question, the answer requires three steps. First, we calculate the price return, then the capital distribution return, and finally, we add these two returns together.

1. Price return

The price return is the change in the price between two different points of time, where P0 denotes the price when the initial investment is made and P1 represents the price at a later valuation date. Here we need the price at the beginning of 2020 and the end of 2020. The price of the bond per EUR100 at the beginning of 2020 is the same as the price at the end of 2019, hence P0 is EUR109.173 and P1 is EUR112.540. Using Equation 1:

$$r_{price} = \frac{P_1 - P_0}{P_0} = \frac{\text{EUR112.540} - \text{EUR109.173}}{\text{EUR109.173}} = \frac{\text{EUR3.367}}{\text{EUR109.173}} = 3.08\%.$$

2. Capital distribution return

The current yield is the capital distribution return—cash flow income through the received coupon payments—denoted Inc and expressed in terms of the initial investment made in the asset. The coupon payment for EUR100 was EUR1.50 for 2020. Substituting into Equation 2:

$$r_{\text{distribution}} = \frac{Inc}{P_0} = \frac{\text{EUR1.50}}{\text{EUR109.173}} = 1.37\%.$$

The current yield is less than the coupon rate, which reflects the fact that the price is above the face value or the principal value of the bond.

3. Total return

The total return encompasses both sources of return. Substituting into Equation 3:

$$r = r_{price} + r_{\text{distribution}} = \frac{P_1 - P_0}{P_0} + \frac{Inc}{P_0}$$

$$= \frac{\text{EUR112.540} - \text{EUR109.173}}{\text{EUR109.173}} + \frac{\text{EUR1.50}}{\text{EUR109.173}}$$

$$= \frac{\text{EUR3.367} + \text{EUR1.50}}{\text{EUR109.173}} = \frac{\text{EUR4.867}}{\text{EUR109.173}} = 4.46\%.$$

For multiple holding periods, aggregating asset returns into a single overall return measure simplifies comparison and understanding. Typically, holding period returns are presented as daily, monthly, or annual figures. To ensure accurate comparisons across different investments and time periods, it's crucial that these returns are calculated using a uniform time period.

Arithmetic and Geometric Holding Period Return

There are different methods for aggregating returns across several holding periods. The simplest is the **arithmetic mean return**, or arithmetic average return, the simple average of all holding period returns. For asset i, where r_{it} is its return during period t and T is the total number of periods, the arithmetic mean return, $\bar{r}_i$, is:

$$\bar{r}_i = \frac{r_{i1} + r_{i2} + \ldots + r_{i,T-1} + r_{iT}}{T}$$

$$= \frac{1}{T}\sum_{t=1}^{T} r_{it}. \quad (4)$$

The geometric mean return, or compound average return, represents the average rate at which an asset's value grows over time, accounting for the compounding of returns. The compounding of returns refers to returns earned on previously earned returns. For asset i, where r_{it} is its return during period t and T is the total number of periods, the geometric mean return, $\bar{r}_{Gi}$, is:

$$\bar{r}_{Gi} = \left[(1 + r_{i1}) \times (1 + r_{i2}) \times \ldots \times (1 + r_{i,T-1}) \times (1 + r_{iT})\right]^{\frac{1}{T}} - 1$$

$$= \sqrt[T]{(1 + r_{i1}) \times (1 + r_{i2}) \times \ldots \times (1 + r_{i,T-1}) \times (1 + r_{iT})} - 1$$

$$= \sqrt[T]{\prod_{t=1}^{T}(1 + r_t)} - 1. \quad (5)$$

Investors can be easily misled by arithmetic returns. For example, if an investment in Year 1 returns 100% and in Year 2 loses 50%, the arithmetic average is:

$$\frac{100\% - 50\%}{2} = \frac{50\%}{2} = 25\%.$$

This solution suggests that, on average, the investment returns 25% per year.

The geometric average is:

$$\sqrt{(1 + 100\%) \times (1 - 50\%)} - 1 = \sqrt{(2.0 \times 0.5)} - 1 = 0\%.$$

The geometric average correctly indicates that over the two years, the net return, taking compounding into account, is 0%, not 25% as suggested by the arithmetic average. This solution demonstrates why the geometric mean is a more accurate reflection of investment performance over time, particularly where annual returns show great variability.

The next case study calculates the arithmetic and geometric performance of Ørsted A/S for 2017–2023.

CASE STUDY

Ørsted A/S Equity Total Returns for 2017–2023

1. Exhibit 7 shows the annual total returns of Ørsted A/S equity for 2017–2023.

Exhibit 7: Ørsted A/S Equity Total Returns for 2017–2023 (in DKK)

	Price (per share in DKK)	Dividend (per share in DKK)	Total return (in %)
31 December 2016*	267.60	---	---
31 December 2017	338.70	6.00	28.81%
31 December 2018	435.70	9.00	31.30%
31 December 2019	689.00	9.75	60.37%
31 December 2020	1,243.50	10.50	82.00%
31 December 2021	835.20	11.50	−31.91%
31 December 2022	631.30	12.50	−22.92%
31 December 2023	374.30	13.50	−38.57%

**The 31 December 2016 price, or year-end price, is used as the starting value for the year 2017.*

Calculate the arithmetic and geometric average annual total return for Ørsted A/S equity for 2017–2023.

Solution

For Ørsted A/S equity, the 2017–2023 arithmetic and geometric average annual total returns are 15.58% and 6.86%, respectively. To calculate the arithmetic mean return, we first compute the sum of the annual returns and then divide it by the number of years.

For Ørsted A/S equity, the 2017–2023 arithmetic mean return, $\bar{r}_i$, is:

$$\bar{r}_i = \frac{r_{i1} + r_{i2} + \ldots + r_{i,T-1} + r_{iT}}{T} = \frac{r_{2017} + r_{2018} + \ldots + r_{2022} + r_{2023}}{7}$$

$$= \frac{28.81\% + 31.30\% + 60.37\% + 82.00\% - 31.91\% - 22.92\% - 38.57\%}{7}.$$

To calculate the geometric average return, we multiply the returns for each period together, take the nth root (where n is the total number of periods;

here, it is 7), and then subtract 1. Hence, for Ørsted A/S equity, the 2017–2023 geometric mean return, $\bar{r}_{Gi}$, is:

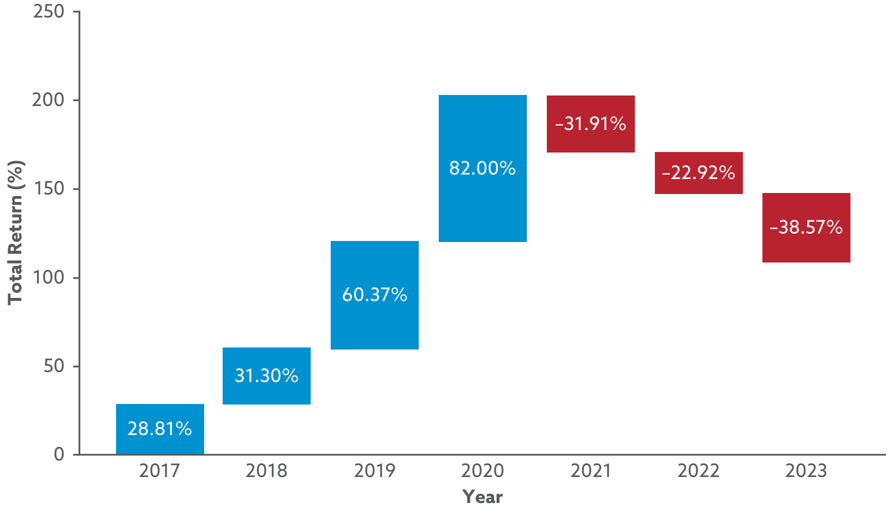
$$\bar{r}_{Gi} = \sqrt[T]{(1 + r_{i1}) \times (1 + r_{i2}) \times \ldots \times (1 + r_{i,T-1}) \times (1 + r_{iT})} - 1$$

$$= \left[(1 + r_{2017}) \times (1 + r_{2018}) \times \ldots \times (1 + r_{2022}) \times (1 + r_{2023})\right]^{\frac{1}{7}} - 1$$

$$= \begin{bmatrix}(1 + 0.2881) \times (1 + 0.3130) \times (1 + 0.6037) \times (1 + 0.8200) \\ \times (1 - 0.3191) \times (1 - 0.2292) \times (1 - 0.3857)\end{bmatrix}^{\frac{1}{7}} - 1$$

The company's equity has initially positive returns, which then turn negative. The trend over the seven years is depicted in Exhibit 8.

Exhibit 8: Ørsted A/S Equity Annual Total Returns, 2017–2023



Typically, both arithmetic and geometric returns are reported. The arithmetic average, or mean return, represents the average rate of return that is the average change in value. As seen in the case study for Ørsted A/S equity, the 2017–2023 arithmetic average annual total return is 15.58%. On average, the stock provided a return of 15.58%, ranging from a high of 82% and a low of –38.57%.

Arithmetic average return calculations assume that the amount invested at the beginning of the period does not change. In an investment portfolio, however, even if there are no cash flows paid into or paid out of the portfolio, the base amount changes due to capital appreciation or depreciation. These impact the initial value of the investment. Accordingly, it is generally more appropriate to consider the geometric return.

The geometric mean return represents the average growth in value or compound rate of return on an investment. This number is of key interest to investors because it quantifies the growth of an investment over multiple periods. For Ørsted A/S equity, the 2017–2023 geometric average annual total return is 6.86%. An investment in Ørsted A/S equity at the beginning of 2017 increased by a total of 59.15% over the seven years to the end of 2023: Each year, the average increase was 6.86%. This value reflects that DKK10,000 invested in Ørsted A/S equity at the start of 2017 has grown in value by DKK5,911.32 at the end of 2023, or:

$$\text{DKK}10{,}000 \times (1 + 6.86\%)^7 - \text{DKK}10{,}000$$

$$= \text{DKK}15{,}911.32 - \text{DKK}10{,}000 = \text{DKK}5{,}911.32$$

The geometric mean has considerable appeal when reporting and comparing historical returns as it is the consistent annual growth rate required to achieve the actual cumulative investment performance. If the annual returns on an investment are constant, the geometric mean will equal the arithmetic mean. Otherwise, the **geometric average return** is always lower. For example, Ørsted A/S equity had an arithmetic annual return of 15.58% between 2017 and 2023, but its geometric annual return was only 6.86%.

This example demonstrates the **arithmetic–geometric mean inequality** principle and highlights the impact of volatility on investment returns. The difference between the arithmetic and geometric mean returns increases with the variability of the returns: The greater the variability of returns, the greater the difference between the arithmetic and geometric average return.

There are two practical points to keep in mind as we conclude this section.

First, the notation for returns can vary and often leads to confusion. This reading explicitly distinguishes between arithmetic and geometric returns, which is not always the case. Different symbols, such as R or even i, might represent returns instead of r. Some authors define $R_{iT} = 1 + r_{iT}$, while other authors define $r_{iT} = 1 + R_{iT}$. Ultimately, the interpretation and calculation of specific returns and variables are context and methodology dependent, which requires the reader to critically assess what is being measured, why, and how.

Second, to calculate annual returns, the prevailing method involves using the closing prices on the final trading day of each period: The total return for 2018 is determined by comparing the price on the last trading day of 2017 with that on the last trading day of 2018. Due to differences in the final trading day of the year across different global capital markets, it is customary to use the closing price of a market's last trading day as the initial price for calculating returns. To calculate monthly returns, the closing price from the end of the previous month is compared with the closing price on the last day of each subsequent month.

Compounding: Daily, Monthly, Quarterly, and Annual Returns

To compare returns across different investments, it is most convenient and common to annualize all available returns. Daily, weekly, monthly, and quarterly returns can all be annualized. Where c is the number of periods in a year, the general equation to annualize returns is:

$$r_{annualized} = (1 + r_{period})^c - 1. \quad (6)$$

A similar approach can be used when monthly or quarterly returns, not annual returns, are needed for comparison purposes. Annual returns can be converted to periodic returns by algebraically rearranging Equation 6.

KNOWLEDGE CHECK: ETF PERFORMANCE

1. An investor evaluating the returns of three recently formed exchange-traded funds gathers the following information in Exhibit 9:

Exhibit 9: ETF Performance Information

ETF	Time since Inception	Return since Inception (%)
1	146 days	4.61
2	5 weeks	1.10
3	15 months	14.35

Assuming that a year has 365 days, or 52 weeks, or 12 months, the ETF with the highest annualized rate of return is:

A. ETF 1.
B. ETF 2.
C. ETF 3.

Solution

The correct response is B. Annualizing returns involves taking the return over a given period, raising it to the power of 1 divided by the fraction of the year that period represents, and then subtracting 1 to obtain the annualized rate. The annualized rate of return for ETF 2 is:

$$12.05\% = 1.0110^{\frac{52}{5}} - 1.$$

This is greater than the annualized rate of ETF 1 and ETF 3.

Answer A is incorrect. The annualized return for ETF 1 is:

$$11.93\% = 1.0461^{\frac{365}{146}} - 1.$$

Answer C is incorrect. The annualized return for ETF 3 is:

$$11.32\% = 1.1435^{\frac{12}{15}} - 1.$$

Despite having the lowest value for the periodic rate, ETF 2 has the highest annualized rate of return because of the reinvestment rate assumption and the compounding of the periodic rate.

2. Annualized returns are calculated by:

A. adding all periodic returns within a year.
B. dividing the periodic returns by the number of periods in a year.
C. raising the sum of 1 plus the geometric average periodic return to the power of the number of periods in a year, then subtracting 1.

Solution

The correct response is C. Annualized returns are determined by raising the sum of 1 plus the periodic return to the power of the number of periods in a year, then subtracting 1, or $r_{annualized} = (1 + r_{period})^c - 1$.

> This approach incorporates the compounding effect of returns over multiple periods within a year.
>
> Answer A is incorrect. The approach fails to account for the compounding effects of returns, which is essential for accurately annualizing returns.
>
> Answer B is incorrect. This approach oversimplifies the calculation and does not accurately capture annualized returns. It neglects the compounding effect, a key aspect in the annualization of returns as seen in the ETF case.

Annualized returns create a consistent metric to evaluate investment performance and predict future returns. They also equate the relationship between cash flows occurring at various times and dates, facilitating the comparison of values over time.

One major limitation of annualizing returns (or any exercise looking at historical performance) is the implicit assumption that earned returns can be repeated—that is, money can be reinvested repeatedly while earning the same return. This assumption is rarely possible and necessitates the oft quoted disclaimer that past performance is not indicative of future results. Furthermore, when applied to returns shorter than one year, such as four months, this process extrapolates short-term performance to the full twelve-month period, ignoring the impact of return variability and potentially overestimating annual performance.

Consider the relatively common occurrence of a stock going up or down by 1% in a given day. Compounding the one-day daily gain of 1% over 250 trading days results in a 1,100% annualized return. A 1% daily loss would be bounded by −100%. Nonetheless, annualization creates a basis for comparison across different investments and different holding periods.

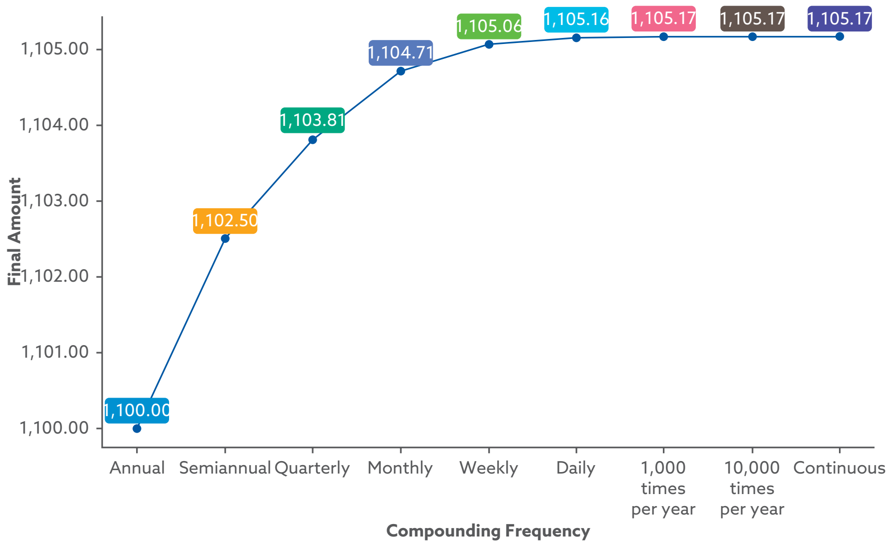
It is important to keep in mind the shortcomings of such an approach. The arithmetic–geometric mean inequality is particularly relevant here, as the variability in returns can significantly impact the compounded (geometric) returns.

Continuous Compounding

An important concept is the **continuously compounded return** associated with a holding period return, which is the natural logarithm of 1 plus that holding period return. This concept is equivalent to the natural logarithm of P_1 over P_0. The terms logarithmic return and continuously compounded return can be used interchangeably.

The more frequently we compound returns, the higher the annual return will be; however, there is a limit. For example, compounding an initial amount of 1,000 at 10% annual rate grows as the following Exhibit 10 shows.

Exhibit 10: The Impact of Compounding



As compounding becomes more frequent, it approaches continuous compounding and the incremental gains from increasing frequency become infinitesimally small. The mathematical limit is captured by the continuous compounding formula, $e^{r \times t}$, which represents the theoretical maximum growth rate for a given interest rate, r, over time, t. e is the base of the natural logarithm and approximately equals 2.71828.

Note that here we are specifically using $\tilde{r}$ to refer to continuously compounded returns, but other textbooks and sources may use a different notation, including r. It is always the reader's responsibility to understand how referenced returns are calculated in each specific situation.

The continuously compounded return from t when the price is P_t to $t + 1$ when the price is P_{t+1}, is:

$$\tilde{r}_{t,\,t+1} = \ln\frac{P_{\mathrm{t+1}}}{P_{\mathrm{t}}} = \ln\left(1 + r_{t,t+1}\right) \quad (7)$$

Because $P_{t+1} = P_t e^{\tilde{r}_{t,\,t+1}}$, the return $\frac{P_T}{P_0}$, where $T > 0$, is the product of the price ratios:

$$\frac{P_T}{P_0} = \frac{P_T}{P_{T-1}} \times \frac{P_{T-1}}{P_{T-2}} \times \frac{P_{T-2}}{P_{T-3}} \times \dots \times \frac{P_2}{P_1} \times \frac{P_1}{P_0}. \quad (8)$$

By taking the natural logarithm of both sides of Equation 8, we get:

$$\ln\left(\frac{P_T}{P_0}\right) = \ln\left(\frac{P_T}{P_{T-1}}\right) + \ln\left(\frac{P_{T-1}}{P_{T-2}}\right) + \ln\left(\frac{P_{T-2}}{P_{T-3}}\right) + \dots + \ln\left(\frac{P_2}{P_1}\right) + \ln\left(\frac{P_1}{P_0}\right).$$

Since each term on the right-hand side represents the continuously compounded return for a specific period, $\tilde{r}_{t-1,t}$, that equation simplifies to the sum of these returns:

$$\ln\left(\frac{P_T}{P_0}\right) = \tilde{r}_{0,1} + \tilde{r}_{1,2} + \tilde{r}_{2,3} \dots + \tilde{r}_{T-2,T-1} + \tilde{r}_{T-1,T}.$$

This implies that the continuously compounded return to time T is the sum of the one-period continuously compounded returns:

$$\tilde{r}_{0,T} = \tilde{r}_{0,1} + \tilde{r}_{1,2} + \tilde{r}_{2,3} \dots + \tilde{r}_{T-2,T-1} + \tilde{r}_{T-1,T} = \sum_{\mathrm{t}=0}^{T} \tilde{r}_t. \quad (9)$$

This additive property of continuously compounded returns provides calculation advantages.

CASE STUDY

Ørsted A/S Continuously Compounded Returns

Exhibit 11 shows the daily prices of the stock from 31 December 2019–31 December 2020 in a Bloomberg screenshot using the ORSTED DC Equity GP <GO> function. The table in Exhibit 12 contains the end of month prices of Ørsted A/S equity.

Exhibit 11: Ørsted A/S Equity Performance, 31 December 2019–30 December 2020 (in DKK)

Source: Bloomberg.

Exhibit 12: Ørsted A/S Equity End-of-Month Prices, 2020 (in DKK)

Date	Price (per share in DKK)
31 December 2019	689.0
31 January 2020	736.0
28 February 2020	695.4
31 March 2020	666.4
30 April 2020	688.4
29 May 2020	786.8
30 June 2020	765.4
31 July 2020	901.0
31 August 2020	882.6
30 September 2020	875.4
30 October 2020	1,014.5
30 November 2020	1,124.0
31 December 2020	1,243.5

Note: The 31 December 2019 price, or year-end price, is used as the starting point for the year 2020.

1. What are the continuously compounded price returns for Ørsted A/S equity for the semi-annual period 1 January 2020–30 June 2020?

   A. 10.51%

   B. 13.64%

   C. 24.54%

   Solution

   The correct response is A. The continuously compounded price return for the period 1 January 2020 (using the price on 31 December, the previous year) through 30 June 2020 is

   $$\widetilde{r_{t,t+1}} = \ln\frac{P_1}{P_0} = \ln\frac{\text{DKK765.40}}{\text{DKK689.00}} = 10.52\,\%.$$

   This is the period return and not the annualized return.

2. What is the annual continuously compounded price return for Ørsted A/S equity from 1 January 2020–31 December 2020?

   A. 54.04%

   B. 59.04%

   C. 80.48%

   Solution

   The correct response is B. The continuously compounded price return for the period 1 January 2020–31 December 2020 is:

   $$\widetilde{r_{t,t+1}} = \ln\frac{P_1}{P_0} = \ln\frac{\text{DKK1,243,50}}{\text{DKK689.00}} = 59.04\%.$$

   Note that adding the continuously compounded price return for the period 1 January 2020–30 June 2020 of 10.51% to the continuously compounded price return for the period 30 June 2020–31 December 2020 of 48.53% yields 59.04%, which is equivalent to the direction calculation. This finding points to a very desirable quality of log returns: They are additive.

3 COMMON RETURN MEASURES

☐ describe, compare, and interpret required rates of return, risk-free rates, risk premia, and inflation

This lesson focuses on common return measures. First, we discuss the minimum rate of return an investor must receive to accept the investment before turning to calculation and interpretation of various commonly used return measures. Then, additional common return metrics are discussed that factor in fees (gross versus net returns), taxes (pre-tax and after-tax returns), inflation (nominal and real returns), and **leverage** (taking on debt to finance an investment).